Advanced Finite Element Formulation for the Convective Wave Equation

Manfred Kaltenbacher
Institute of Mechanics and Mechatronics, TU Wien, Vienna, Austria.

Andreas Hüppe
Institute of Mechanics and Mechatronics, TU Wien, Vienna, Austria. (now at ANSYS Germany)

Summary

The convective wave equation models wave propagation in flowing media by including convection and refraction effects. Thereby, the primary physical quantity is the scalar acoustic potential, which is related to the acoustic pressure by the substantial derivative w.r.t time scaled by the mean density of the medium. A standard Finite Element (FE) formulation of this partial differential equation results in spurious modes and may even become unstable. Therefore, we present a FE formulation, which preserves the skew symmetry of the convective wave operator resulting in a stable computational scheme independent of the Mach number. The properties are studied by performing an eigenvalue analysis.

1. Introduction

The convective wave equation (CWE) describes acoustic wave propagation in flowing media. A systematic derivation of CWE has been provided in [1] starting at the full set of compressible flow equations and applying a perturbation ansatz. The final partial differential equation is similar to the standard wave equation, but instead of the second order partial time derivative it has a second order substantial time derivative. Furthermore, in computational aeroacoustics, the reformulation of the acoustic perturbation equations (see [2]) leads to the same convective wave operator and the substantial derivative of the incompressible flow pressure as a source term. This partial differential equation has again the scalar acoustic potential as the search for quantity and has been named perturbed convective wave equation (PCWE) [3]. Thereby, it can be shown that a standard Finite-Element (FE) formulation leads to eigenvalues with a strong real part. Furthermore, the magnitude of the eigenvalues scale with the Mach number of the flow.

In our contribution, we show the derivation of a stable FE formulation by applying an appropriate transformation of the standard FE formulation. Thereby, we follow ideas presented in [5], where a stable FE formulation for the acoustic perturbation equations, which are based on linearized mass and momentum conservation including convective operators and solving for the acoustic pressure and particle velocity, has been presented. Furthermore, we demonstrate by an eigenvalue analysis the strong reduction of the real part of spurious eigenvalues, and thus the stability of the computational scheme.

2. Finite Element Formulation

We consider the following homogeneous convective wave equation

$$\frac{1}{c_0^2}\frac{\mathrm{D}^2\psi}{\mathrm{D}t^2} - \nabla\cdot\nabla\psi = 0 \tag{1}$$

as derived in [1] for the case of sound in fluids with unsteady inhomogeneous flow as well as in [3] for modeling aeroacoustics phenomena. In (1) ψ denotes the scalar acoustic potential, and c_0 the speed of sound in the medium. The substantial derivative $\mathrm{D}/\mathrm{D}t$ computes by

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \overline{\boldsymbol{u}}\cdot\nabla \tag{2}$$

with $\overline{\boldsymbol{u}}$ the mean flow velocity (constant in time). In a next step, we derive the weak formulation. In doing so, we introduce an appropriate test function φ, multiply (1) by it and integrate over the whole computational domain Ω

$$\frac{1}{c_0^2}\int_\Omega \varphi\,\frac{\mathrm{D}^2\psi}{\mathrm{D}t^2}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \varphi\,\nabla\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x} = 0\,. \tag{3}$$

Expanding the substantial derivative in (3) results in

$$\underbrace{\frac{1}{c_0^2}\int_\Omega \varphi\,\frac{\partial^2\psi}{\partial t^2}\,\mathrm{d}\boldsymbol{x}}_{I}+\underbrace{\frac{1}{c_0^2}\int_\Omega \varphi\,(\overline{\boldsymbol{u}}\cdot\nabla)\,\frac{\partial\psi}{\partial t}\,\mathrm{d}\boldsymbol{x}}_{IIa}$$
$$+\underbrace{\frac{1}{c_0^2}\int_\Omega \varphi\frac{\partial}{\partial t}(\overline{\boldsymbol{u}}\cdot\nabla\,\psi)\,\mathrm{d}\boldsymbol{x}}_{IIb}$$
$$+\underbrace{\frac{1}{c_0^2}\int_\Omega \varphi\,(\overline{\boldsymbol{u}}\cdot\nabla)(\overline{\boldsymbol{u}}\cdot\nabla)\,\psi\,\mathrm{d}\boldsymbol{x}}_{III}$$
$$+\underbrace{\int_\Omega \varphi\,\nabla\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x}}_{IV}=0\,. \tag{4}$$

The term I is a standard bilinear form and needs no special treatment. However, the terms IIa and IIb are subjected to the mean flow field, which is not necessarily homogeneous. By exploring the property that the mean flow is time-independent, the two terms IIa and IIb in (4) may be combined to

$$\frac{2}{c_0^2}\int_\Omega \varphi\,(\overline{\boldsymbol{u}}\cdot\nabla)\,\frac{\partial\psi}{\partial t}\,\mathrm{d}\boldsymbol{x}\,. \tag{5}$$

The analysis of this term reveals that it has only first order derivatives w.r.t space and therefore an integration by parts is not necessarily needed. However, as pointed out in [4], the skew symmetry of the operator has to be preserved also in the discrete form (obtained after space discretization, e.g., with the FE method), in order to achieve energy conservation and stability. The eigenvalue analysis in Sec. 3 will demonstrate that spurious modes arise and some of them even have a strong positive real part, so that unstable computations are observed. Therefore, we follow the ideas in [5] and first rewrite (5) by

$$\frac{2}{c_0^2}\int_\Omega \varphi\,\nabla\cdot\left(\overline{\boldsymbol{u}}\,\frac{\partial\psi}{\partial t}\right)\,\mathrm{d}\boldsymbol{x}\,. \tag{6}$$

Here, the derivation just holds for the case that $\overline{\boldsymbol{u}}$ is solenoidal or in an FE setting, where $\overline{\boldsymbol{u}}$ is piecewise constant for each finite element. Now, an integration by parts can be performed to ensure skew symmetry, and results in

$$\begin{aligned}\int_\Omega \varphi\,\nabla\cdot\left(\overline{\boldsymbol{u}}\,\frac{\partial\psi}{\partial t}\right)\,\mathrm{d}\boldsymbol{x} &= -\int_\Omega \nabla\cdot(\varphi\,\overline{\boldsymbol{u}})\,\frac{\partial\psi}{\partial t}\,\mathrm{d}\boldsymbol{x}\\ &\quad+\int_\Gamma \varphi\,(\overline{\boldsymbol{u}}\cdot\boldsymbol{n})\,\frac{\partial\psi}{\partial t}\,\mathrm{d}\boldsymbol{s}\\ &= -\int_\Omega (\overline{\boldsymbol{u}}\cdot\nabla)\,\varphi\,\frac{\partial\psi}{\partial t}\,\mathrm{d}\boldsymbol{x}\\ &\quad+\int_\Gamma \varphi\,(\overline{\boldsymbol{u}}\cdot\boldsymbol{n})\,\frac{\partial\psi}{\partial t}\,\mathrm{d}\boldsymbol{s}\end{aligned} \tag{7}$$

Exploring this result, we may rewrite (5) using (7) by

$$\begin{aligned}&-\frac{1}{c_0^2}\int_\Omega (\overline{\boldsymbol{u}}\cdot\nabla)\,\varphi\,\frac{\partial\psi}{\partial t}\,\mathrm{d}\boldsymbol{x}\\ &+\frac{1}{c_0^2}\int_\Gamma \varphi\,(\overline{\boldsymbol{u}}\cdot\boldsymbol{n})\,\frac{\partial\psi}{\partial t}\,\mathrm{d}\boldsymbol{s}\\ &+\frac{1}{c_0^2}\int_\Omega \varphi\,(\overline{\boldsymbol{u}}\cdot\nabla)\,\frac{\partial\psi}{\partial t}\,\mathrm{d}\boldsymbol{x}\,,\end{aligned} \tag{8}$$

which will guarantee skew symmetry also in the space discrete form. The surface integral may vanish for no-slip boundary conditions in a computational fluid dynamics computation.

The term III is also integrated by parts, and for the same assumption towards the mean flow velocity $\overline{\boldsymbol{u}}$ as before, we arrive at

$$\begin{aligned}&\frac{1}{c_0^2}\int_\Omega \varphi\,(\overline{\boldsymbol{u}}\cdot\nabla)(\overline{\boldsymbol{u}}\cdot\nabla)\,\psi\,\mathrm{d}\boldsymbol{x}\\ &=\frac{1}{c_0^2}\int_\Omega \nabla(\overline{\boldsymbol{u}}\varphi)\,(\overline{\boldsymbol{u}}\cdot\nabla)\,\psi\,\mathrm{d}\boldsymbol{x}\\ &=-\frac{1}{c_0^2}\int_\Omega (\overline{\boldsymbol{u}}\cdot\nabla)\,\varphi\,(\overline{\boldsymbol{u}}\cdot\nabla)\,\psi\,\mathrm{d}\boldsymbol{x}\\ &\quad+\frac{1}{c_0^2}\int_\Gamma \varphi\,(\overline{\boldsymbol{u}}\cdot\boldsymbol{n})\,(\overline{\boldsymbol{u}}\cdot\nabla)\psi\,\mathrm{d}\boldsymbol{s}\,.\end{aligned} \tag{9}$$

The term IV in (4) is treated as usual and results in

$$\begin{aligned}\int_\Omega \varphi\,\nabla\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x} &= -\int_\Omega \nabla\varphi\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x}\\ &\quad+\int_\Gamma \varphi\,\boldsymbol{n}\cdot\nabla\psi\,\mathrm{d}\boldsymbol{s}\,.\end{aligned} \tag{10}$$

Applying a standard FE ansatz with appropriate FE basis functions $N_i(\boldsymbol{x})$

$$\begin{aligned}\varphi\approx\varphi^h &= \sum_a N_a(\boldsymbol{x})\,\varphi_a(t)\\ \psi\approx\psi^h &= \sum_b N_b\boldsymbol{x})\,\psi_b(t)\end{aligned} \tag{11}$$

results in the following semi-discrete Galerkin formulation

$$\boldsymbol{M}\underline{\ddot{\psi}}^h+\boldsymbol{C}\underline{\dot{\psi}}^h+\boldsymbol{K}\underline{\psi}^h=\underline{f}^h\,. \tag{12}$$

In (12) $\underline{\psi}^h$ is an algebraic vector collecting all the unknowns of the scalar acoustic potential, a dot over a variable denotes the derivative with respect to time, i.e. $\partial^2\underline{\psi}^h/\partial t^2=\underline{\ddot{\psi}}^h$, and $\underline{f}^h$ the right hand side according to a given source term or boundary conditions. Furthermore, in (12) $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}\in\mathbb{R}^{n_{\mathrm{eq}}}\times\mathbb{R}^{n_{\mathrm{eq}}}$ are the mass, damping and stiffness matrices with n_{eq}

the number of unknowns, whose entries compute as follows

$$M_{ab} = \frac{1}{c_0^2} \int\limits_{\Omega} N_a \, N_b \, \mathrm{d}\boldsymbol{x} \qquad (13)$$

$$\begin{aligned} C_{ab} = & -\frac{1}{c_0^2} \int\limits_{\Omega} \Big((\overline{\boldsymbol{u}} \cdot \nabla) \, N_a \Big) \, N_b \, \mathrm{d}\boldsymbol{x} \\ & +\frac{1}{c_0^2} \int\limits_{\Gamma} N_a \, (\overline{\boldsymbol{u}} \cdot \boldsymbol{n}) \, N_b \, \mathrm{d}\boldsymbol{s} \\ & +\frac{1}{c_0^2} \int\limits_{\Omega} N_a \, (\overline{\boldsymbol{u}} \cdot \nabla) \, N_b \, \mathrm{d}\boldsymbol{x} \\ & -\frac{1}{c_0^2} \int\limits_{\Omega} (\overline{\boldsymbol{u}} \cdot \nabla) \, N_a \, (\overline{\boldsymbol{u}} \cdot \nabla) \, N_b \, \mathrm{d}\boldsymbol{x} \\ & +\frac{1}{c_0^2} \int\limits_{\Gamma} N_a \, (\overline{\boldsymbol{u}} \cdot \boldsymbol{n}) \, (\overline{\boldsymbol{u}} \cdot \nabla) N_b \, \mathrm{d}\boldsymbol{s} \end{aligned} \qquad (14)$$

$$K_{ab} = \int\limits_{\Omega} \nabla N_a \cdot \nabla N_b \, \mathrm{d}\boldsymbol{x} \, . \qquad (15)$$

3. Eigenvalue Analysis

For the investigation of spurious modes, we perform computations for plane waves in a channel of length L with uniform background flow. As boundary conditions, we set $\psi(t, x = 0) = 0$ and $\psi(t, x = L) = 0$. For this case, the convective wave equation (1) may be rewritten as [6]

$$\frac{\partial^2 \psi}{\partial t^2} + 2 M c_0 \frac{\partial^2 \psi}{\partial x \partial t} + (M^2 - 1) c_0^2 \frac{\partial^2 \psi}{\partial x^2} = 0 \, . \qquad (16)$$

In (16) M denotes the Mach number computed by

$$M = \frac{|\overline{\boldsymbol{u}}|}{c_0} \, .$$

For a plane wave, one can make the ansatz

$$\psi(x,t) = \hat{\psi} \, e^{j(kx - \omega t)}$$

with k the the wave number, ω the angular frequency and j the complex unit. Substituting this ansatz into (16) results in the dispersion relation

$$-\omega^2 + 2\omega M \, c_0 \, k - (M^2 - 1) \, c_0^2 \, k^2 = 0 \, . \qquad (17)$$

The solution for k by fixed ω reads as

$$k_{1,2} = -\frac{\omega \big(M \, c_0 \mp c_0 \big)}{\big(M^2 \, c_0^2 - c_0^2 \big)} = -\frac{\omega}{(M \pm 1) \, c_0} \, . \qquad (18)$$

The general solution for the one-dimensional waves travelling within the channel are given by

$$\psi(x) = A_1 e^{j k_1 x} + A_2 e^{-j k_1 x} \, . \qquad (19)$$

By incorporation of the boundary conditions, one obtains the eigenvalues analytically by

$$\omega_n = \frac{c_0 \pi n}{L} (1 - M^2) \, . \qquad (20)$$

The discrete eigenvalues are obtained by using the ansatz

$$\underline{\Psi}^h = \Psi \, e^{st} \, ; \; s = j\omega$$

and substituting it into (12), which results in

$$\big(s^2 \boldsymbol{M} + s \boldsymbol{C} + \boldsymbol{K} \big) \; \Psi = \boldsymbol{0} \, . \qquad (21)$$

This equation is satisfied by the i-th latent root s_i, and i-th latent vector $\underline{\Psi}$ of the λ-matrix problem [7], so that

$$s_i^2 \boldsymbol{M} \underline{\Psi}_i + s_i \boldsymbol{C} \underline{\Psi}_i + \boldsymbol{K} \underline{\Psi}_i = \boldsymbol{0} \quad \forall_i \in 1..n_{\mathrm{eq}} \, . \qquad (22)$$

For the numerical computation, we have computed the matrices by our in-house research software CFS++ [8] and then applied Matlab using the function *polyeig*. Homogeneous Neumann boundary conditions on the horizontal boundaries Γ_{N} and homogeneous Dirichlet on the vertical boundaries Γ_{D} are used (see Fig. 1). The first smallest non zero eigenvalues should correspond with the frequencies of the one dimensional channel, i.e. $\lambda \simeq j\omega_n$ (higher values of λ correspond to eigenfunctions oscillating in the height H of the channel). Thereby, the discrete eigenvalues can be interpreted as follows:

- If there are eigenvalues λ which do not coincide with the analytical ones, we can say that these are spurious modes.
- If the discrete eigenvalues have a positive real part, the solution becomes unstable.
- Spurious modes with a large negative real part are quickly damped and do not disturb the solution.

In this sense, we distinguish not only between physical and spurious modes but also between *good* and *bad* modes depending on the sign of their real part. For

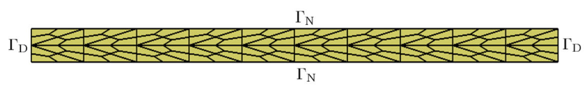

Figure 1. Pseudo one dimensional channel with strongly distorted elements.

the numerical evaluation of the discrete eigenvalues a long and thin channel with a length to width ratio of 20 : 1 is chosen. The channel is discretized with distorted quadrilateral elements as displayed in Fig. 1.

Initially the flow velocity is set to zero, which means that the eigenvalues of the standard wave equation are computed (no convective terms). The result is displayed in Fig. 2, and the discrete eigenvalues are all at the imaginary axis as indicated by the markers. All discrete eigenvalues have zero real part and coincide with the analytical ones. In contrast to the stable

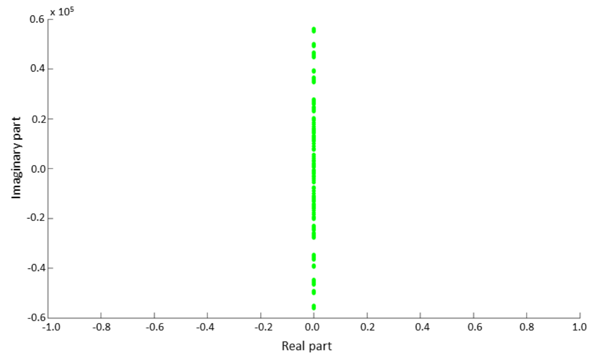

Figure 2. Discrete eigenvalues for zero Mach number (standard wave equation).

computations with zero Mach number, the situation looks different when a constant mean flow of Mach numbers 0.1, 0.2, 0.3 is present. Now, the eigenvalues of the convective wave equation as discussed in Sec. 2 has to be computed. In a first step, we use the standard formulation, which does not perform an integration by parts for terms IIa, IIb. Therefore, the skew symmetry is not preserved at the discrete level. As demonstrated by Fig. 3, a lot of spurious modes are added to the system. Even more dramatically, spurious eigenvalues with a strong positive real part occur which are responsible for unstable computations. Furthermore, Fig. 3 demonstrates that the positive

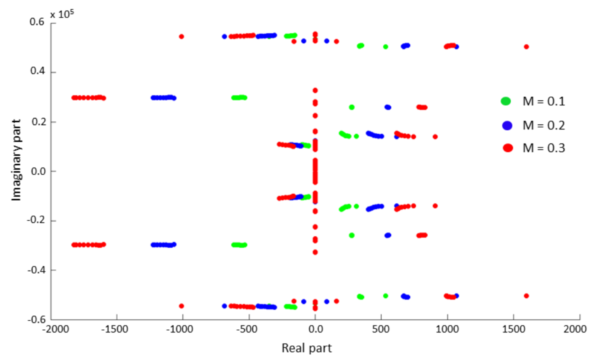

Figure 3. Discrete eigenvalues for different Mach numbers for standard formulation.

values of the spurious eigenvalues strongly increase with the Mach number. In a second step, we perform the computation of the advanced formulation, for which an integration by parts is performed for the terms IIa, IIb. Thereby, the skew symmetry is preserved also at the discrete level. Figure 4 displays the computed eigenvalues, and as one can see, the real part of the spurious eigenvalues have been reduced by several magnitudes, and do not increase with higher Mach numbers. Thereby, the stability of this advanced

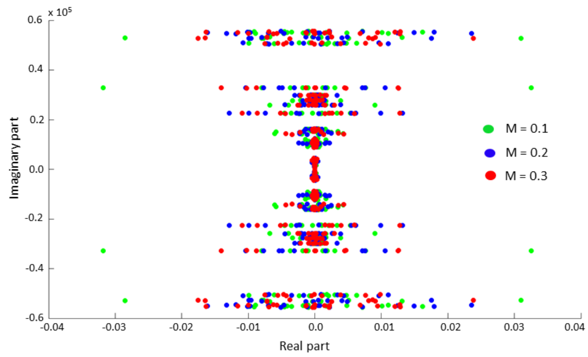

Figure 4. Discrete eigenvalues for different Mach numbers for advanced formulation.

formulation is demonstrated.

4. CONCLUSIONS

The convective wave equation takes into account convection and refraction effects of waves propagating through flowing media. A standard FE formulation results in spurious modes and for increasing Mach number the numerical solution may become unstable. The introduced advanced FE formulation preserves the skew symmetry of the convective wave operator even at the discrete level, and therefore achieves energy conservation and stability. Successful application to real-life problems can be found, e.g., in [3], where the flow induced sound of an axial fan has been computed.

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