Analysis of Characteristics of Elastic Waves Propagating on a Vibrating Cylindrical Shell at Frequencies around a Ring Frequency

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Summary
The vibration of a cylindrical shell is generated due to elastic waves propagating on the shell. The wave propagation is governed by a dispersion relation. The assumption of a thin shell allows the dispersion relation to be separated into three relations related to the propagation of flexural waves and two types of membrane waves. Above the ring frequency, those waves are clearly identified as flexural, shear and longitudinal waves. Below the ring frequency, the characteristics of those waves are identified with dependency of the direction of wave propagation. In this paper the dispersion relations are obtained theoretically and experimentally at two frequencies above and below the ring frequency. The experimental results of the dispersion relations have been obtained by using wavenumber analysis of in-plane surface vibration data obtained experimentally on a point-excited cylindrical shell. Those are compared with the theoretical results. The obtained dispersion relations are analysed to identify the characteristics of waves propagating on the vibrating cylindrical shell at frequencies around the ring frequency.

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1. Introduction

The vibration of a cylindrical shell is generated due to elastic wave propagation on the shell. In order to understand vibration characteristics of the shell, it is necessary to understand the characteristics of elastic wave propagation on the shell. The basic idea [1] was introduced that waves propagating on a point-excited cylindrical shell behave like waves propagating in a two-dimensional unbounded homogeneous anisotropic medium with excitation forces that are periodic in the transverse direction. The idea was used the to analyze the wave propagation on the cylindrical shell in [2]. The numerical results yielded the implication that major features of the thin plate model's prediction could hold relatively well, down to frequencies as low as twice the ring frequency of the cylindrical shell. The wave propagation on a cylindrical shell well below the ring frequency including a fluid-loading was examined in [3]. The general frequency characteristics of dispersion relation of elastic waves propagating on a vibrating cylindrical shell was studied in [4].

There is a significant change of characteristics of waves propagating on a cylindrical shell around the ring frequency. In this paper, the characteristics of elastic waves propagation on the shell around the ring frequency has been analyzed theoretically and experimentally. The assumption of a thin shell allows the dispersion relation to be separated into three relations related to the propagation of flexural waves and two types of membrane waves. Above the ring frequency, those waves are clearly identified as flexural, shear and longitudinal waves. Below the ring frequency, the characteristics of those waves are identified with dependency of the direction of wave propagation. The dispersion curves are obtained theoretically and experimentally at two frequencies above and below the ring frequencies. Those correspond to two and three curves below and above the ring frequency, respectively. The theoretical and experimental results of the dispersion curves are compared each other and are analysed to identify the characteristics of waves propagating on the vibrating cylindrical shell at frequencies around the ring frequency.

2. Theoretical Analysis

2.1 Dynamic equation and dispersion relation

The vibration of the cylindrical shell with nominal radius a and thickness h is described in terms of displacement vector **V** with components of u, v and w in radial (r), circumferential (φ) and axial (z) directions in Figure 1. The displacement field can be described using Donnell's thin shell dynamic equation [5] that takes the form as

$$\mathbf{L}\,\mathbf{V} = \mathbf{F} \qquad (1)$$

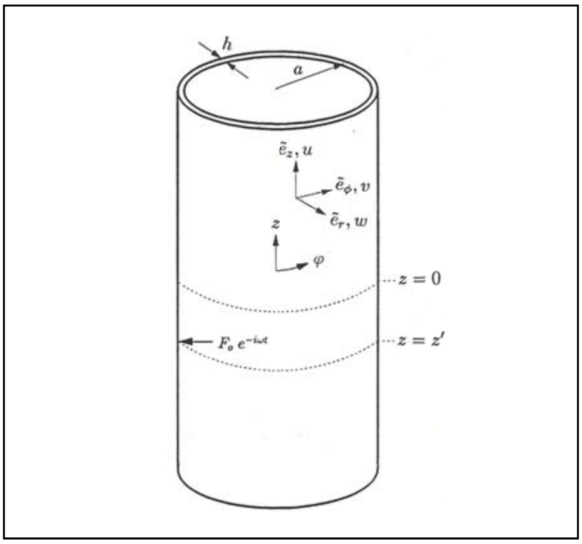

Figure 1. Cylindrical shell excited by a point force

where **L** is a linear operator [6]. **F** denotes the external force vector applied at $z = z'$. The exciting force at angular frequency ω generates vibration of the shell. The vibration of the shell is generated due to propagation of elastic wave. This physical idea allows the response to be taken as spatial Fourier integral [7]. The axial component of displacement vector, for example, with the supressed time dependent $e^{-i\omega t}$, is expressed as

$$\mathrm{u}(\varphi, \mathrm{z}, \omega) = \sum_{=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} U_n\,(k_z, \omega)\, e^{in\varphi} e^{ik_z}\, dk_z \qquad (2)$$

where k_z and n correspond to the axial wavenumber and the circumferential mode as $n = k_\varphi\, a$ with the circumferential wavenumber k_φ, respectively. The Fourier transform U_n corresponds to the complex amplitude of the axial displacement component, that is related to waves propagating in the direction of the wavenumber vector with the angle θ_k as

$$\mathbf{k} = k_\varphi \tilde{e}_\varphi + k_z \tilde{e}_z = k\, sin\theta_k\, \tilde{e}_\varphi + k\, cos\theta_k \tilde{e}_z. \qquad (3)$$

Here $\tilde{e}_\varphi$ and $\tilde{e}_z$ denote the unit vector in the corresponding direction. The algebraic equations governing the complex amplitudes U_n, V_n and W_n are obtained from equation 1 as

$$[L_n] \begin{Bmatrix} U_n \\ V_n \\ W_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{zn} \end{Bmatrix} \qquad (4)$$

where $[L_n]$ is the linear operator matrix obtained by applying the Fourier integral into the equation 1. F_{zn} denotes the Fourier transform of the radial point force.

For a nontrivial solution of equation 4, the determinant of the coefficient matrix should be zero as

$$\begin{aligned} \det([L]) &= D(n, k_z, \omega) \\ &= (k^4\epsilon - \Omega^2)(\Omega^2 - \, k^2)\left(\Omega^2 - \frac{1-\nu}{2}k^2\right) \\ &\quad + \frac{1-\nu}{2}\{(1-\nu^2)k^4 cos^4\theta_k \\ &\quad - (2\nu+3)k^2 cos^2\theta_k \Omega^2 \\ &\quad - \, k^2 sin^2\theta_k \Omega^2\} + \Omega^4 \; = \; 0. \end{aligned} \qquad (5)$$

Here the dimensionless frequency $\Omega = \omega/\omega_r$ where $\omega_r = (1/a)\{E/\rho(1-\nu^2)\}^{1/2}$ denotes the ring frequency of the cylinder. E, ρ and ν correspond to Young's modulus, density and Poisson's ratio, respectively. Equation 5 is called the dispersion relation which governs the wave propagation on the vibrating cylindrical shell. Here it is also called the exact dispersion relation.

2.2 Analysis of dispersion relation

Wave propagation of the waves is governed by the dispersion relation in equation 5. The waves consist of propagating waves and exponentially decaying waves, that are determined with dependency on nature of roots of the dispersion relation. Propagating waves include flexural and two types of membrane waves.

For thin shells with $h/a \leq 1/20$ (or $\epsilon = h^2/(12a^2 \leq 2.1 \times 10^{-4}$)[8], the function D is factored [2,4] to

$$D \simeq D_{flex} D_{mem} = D_{flex} D_{mem1} D_{mem2} \qquad (6)$$

where

$$D_{flex} = \epsilon {k_0}^4 - \Omega^2 + (1-\nu^2) cos^4\theta_k \qquad (7)$$

$$D_{mem} = \frac{1-\nu}{2} D_{mem1} D_{mem2}. \quad (8)$$

Here,

$$D_{mem1} = \left(H_1 - {k_0}^2\right);\ D_{mem2} = \left(H_2 - {k_0}^2\right) \quad (9)$$

where

$$H_1 = \frac{I_4 + \sqrt{I_8}}{I_2}\ ; H_{12} = \frac{I_4 - \sqrt{I_8}}{I_2} \quad (10)$$

$$I_2 = (1-\nu)\{(1-\nu^2)cos^4\theta_k - \Omega^2\}$$

$$I_4 = -\frac{3-\nu}{2}\Omega^4 + \frac{1-\nu}{2}\{1 + 2(1+\nu)cos^2\theta_k\}\Omega^2$$

$$I_8 = \frac{1+\nu^2}{4}\Omega^8 + \frac{1-\nu^2}{2}p\Omega^6 + \frac{1-\nu^2}{4}q\Omega^4$$

$$p = 1 - 2(3-\nu)cos^2\theta_k + 4(1-\nu)cos^4\theta_k$$

$$q = 1 + 4(1+\nu)cos^2\theta_k - 4(1-\nu^2)cos^4\theta_k\ .$$

Here $k_0 = ka$ corresponds to the dimensionless wavenumber. $D_{flex} = 0$ gives the dispersion relation for flexural waves. The factor D_{mem} is independent of the parameter ϵ, i.e. thickness h. $D_{mem1} = 0$, $D_{mem2} = 0$ correspond to the dispersion relations of the different type of membrane waves.

Considering the high frequency limit as $\Omega \gg 1$, the factor can be approximated as

$$D_{flex} \simeq \epsilon {k_0}^4 - \Omega^2 \quad (11)$$

$$D_{mem1} \simeq \Omega^2 - {k_0}^2 = D_{long} \quad (12)$$

$$D_{mem2} \simeq \frac{2}{1-\nu}\left(\Omega^2 - \frac{1-\nu}{2}{k_0}^2\right), = D_{shear}. \quad (13)$$

$D_{flex} = 0$, $D_{long} = 0$ and $D_{shear} = 0$ correspond to the dispersion relation of flexural, longitudinal and shear waves, respectively. Those can be approximated for moderate values of Ω as

$$D_{long} = \Omega^2 - {k_0}^2 - (sin^2\theta_k + \nu cos^2\theta_k)^2 \quad (14)$$

$$D_{shear} = \Omega^2 - \frac{1-\nu}{2}{k_0}^2 - 2(1-\nu)sin^2\theta_k cos^2\theta_k \quad (15)$$

For waves propagating in the axial direction ($\theta_k = 0$), D_{mem1} and D_{mem2} in equation 9 take the forms [4]

$$D_{mem1} = \frac{2}{1-\nu}\left(\Omega^2 - \frac{1-\nu}{2}{k_0}^2\right) \quad (16)$$

$$D_{mem2} = \frac{\Omega^2(\Omega^2 - 1)}{\Omega^2 - (1-\nu^2)} - {k_0}^2 \quad (17)$$

at $\Omega < \Omega_m$ where

$$\Omega_m = \{\frac{(1-\nu)(1+2\nu)}{1+\nu}\}^{1/2} \quad (18)$$

which corresponds to $\Omega_m = 0.94$ with $\nu = 0.28$. But at $\Omega \geq \Omega_m$ those take the opposite forms as

$$D_{mem1} = \frac{\Omega^2(\Omega^2 - 1)}{\Omega^2 - (1-\nu^2)} - {k_0}^2 \quad (19)$$

$$D_{mem2} = \frac{2}{1-\nu}\left(\Omega^2 - \frac{1-\nu}{2}{k_0}^2\right) \quad (20)$$

Here $D_{mem1} = 0$ in equation (16) and $D_{mem2} = 0$ in equation (20) correspond to the dispersion relation of plates. Therefore, for waves propagating in the axial direction ($\theta_k = 0$) at $\Omega < \Omega_m$, D_{mem1} and D_{mem2} are related to the dispersion relation of shear and longitudinal waves, respectively. However those are related to longitudinal and shear waves, respectively, at $\Omega \geq \Omega_m$. In low frequency limit ($\Omega \ll 1$), D_{mem2} in equation 17 is approximated by

$$D_{mem2} = \frac{\Omega^2}{1-\nu^2} - {k_0}^2 \quad (21)$$

It indicates that the waves, being represented by $D_{mem2} = 0$ and propagating in the axial direction in low frequency limit, correspond to the longitudinal waves with the phase velocity $(E/\rho)^{1/2}$ corresponding to so-called “bar-velocity”.

For waves propagating in the circumferential direction ($\theta_k = \pi/2$), D_{mem1} and D_{mem2} in equation 9 take the forms

$$D_{mem1} = (\Omega^2 - 1) - {k_0}^2 \quad (22)$$

$$D_{mem2} = \frac{2}{1-\nu}\left(\Omega^2 - \frac{1-\nu}{2}{k_0}^2\right) \quad (23)$$

D_{mem1} and D_{mem2} are associated with the dispersion relations to longitudinal and shear waves, respectively. The more analysis about the dispersion relation can be referred in [4, 9].

3. Experimental Analysis

Fourier transform U_n in equation 2, i.e. the complex amplitude of waves propagating on the cylindrical shell, can be expressed in terms of the axial displacement component u as

$$U_n(k_z, \omega) = \sum_{=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\varphi, z, \omega)\, e^{ik_z z} e^{in\varphi} d\varphi dz \quad (24)$$

It means that the amplitudes of waves propagating in the direction of wavenumber vector $\mathbf{k} = k_\varphi \tilde{e}_\varphi + k_z \tilde{e}_z$ can be evaluated if the vibration field data is experimentally measured on the cylindrical shell and is processed with equation 24.

The experimental model is a finite cylindrical shell made of stainless steel. The dimensions of the shell are: *a=0.076m, h=1.5mm, L=0.93m*. In order to approximate the free-free boundary conditions at both ends of the shell, the shell was held between two aluminium end caps with four uniformly spaced pieces of neoprene along each end of the shell as shown in Figure 2. The shell was excited by a piezoelectric shaker located inside of the shell and *0.32L* above the bottom of the shell. The shaker was driven at *18,275 Hz* $(= 1.65\,\Omega)$ and 9,238 Hz $(= 0.84\,\Omega)$. The scanning laser Doppler vibrometry system was used to measure the in-plane vibration of the shell. The detailed description about the experimental system can be found in [9,10]. Two different wave fields were generated on the regions of the shell above and below the excitation point. The axial displacement was measured at *32×32* points on the surface of the shell above the excitation point.

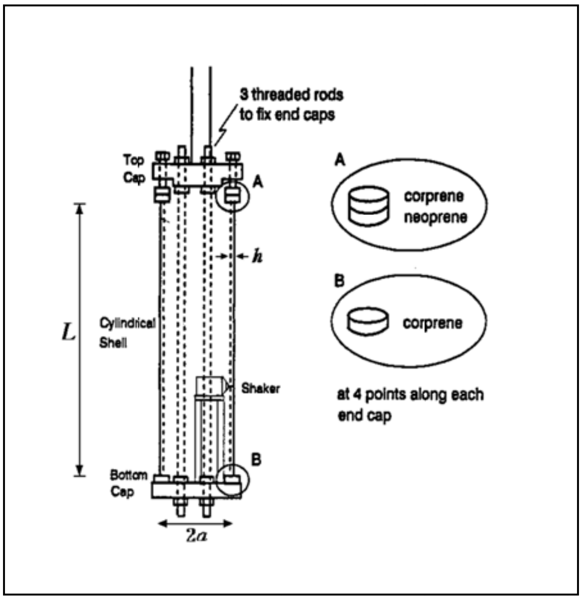

Figure 2. Point-excited experimental cylindrical shell

The data of experimental axial displacement on the shell was processed by means of spatial FFT using equation 24. The magnitude of the corresponding Fourier transform U_n can be expressed in a three-dimensional picture for the wavenumber region. This result is called a wave spectrum. From the wave spectrum, we can find the magnitude and types of waves occurring at the excitation frequency, and also can estimate the dispersion relations of waves.

4. Results

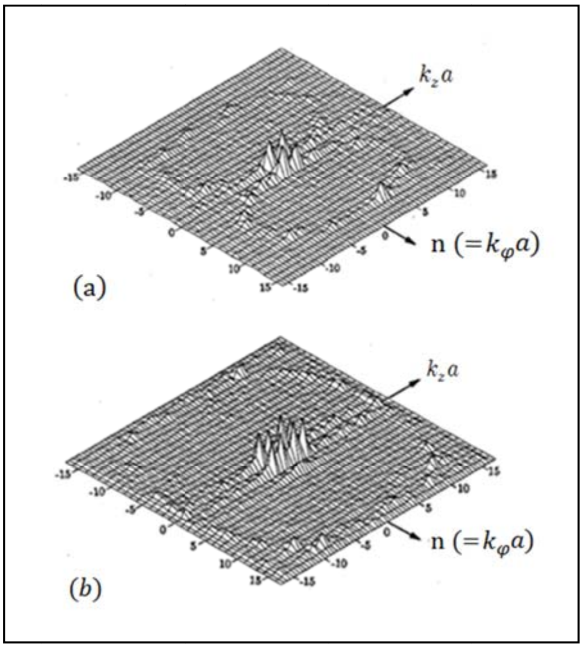

Figure 3. Wave spectrum of the experimental axial displacement fields measured at (a) *9,238 Hz* $(= 0.84\,\Omega)$ and (b)*18,275 Hz* $(= 1.65\,\Omega)$.

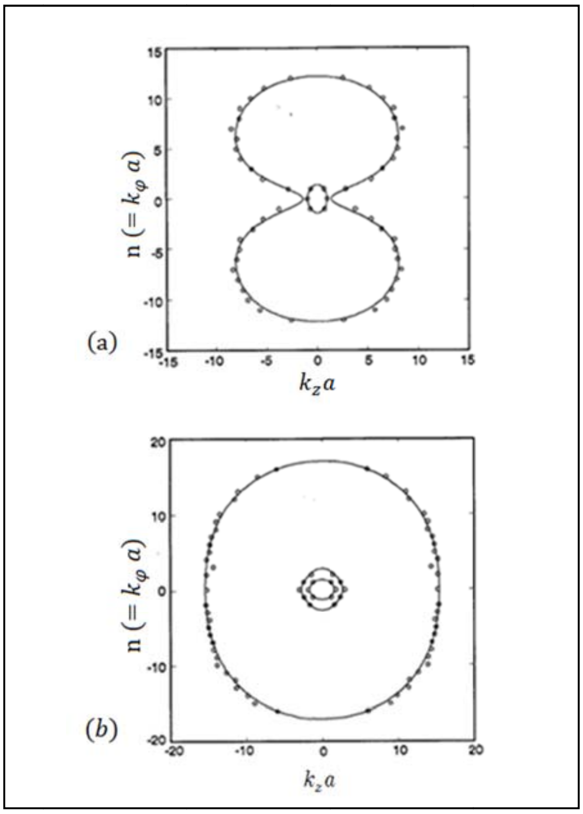

Figure 4. Comparison between the wave number reconstruction from the experimental data (o) and results from the theoretical exact dispersion relation (-): (a) *9,238 Hz* $(= 0.84\,\Omega)$ and (b)*18,275 Hz* $(= 1.65\,\Omega)$.

The dispersion curves represented by the dispersion relation can be evaluated from the peaks in the wave spectra above and below the ring frequency in Figure 3. Figure 4 shows comparison of the experimental result for the dispersion curves with the theoretical prediction for the exact dispersion relation in equation 5. It shows good agreement between the measured and predicted dispersion curves. The dispersion curve in Figure 4(a) shows one of low frequency characteristics of a figure-8 shaped dispersion curve below the ring frequency ($\Omega = 1$). The dispersion curves in Figure 4(b) show that one of high frequency characteristics of circle shaped dispersion curves that represent waves propagating on a plate.

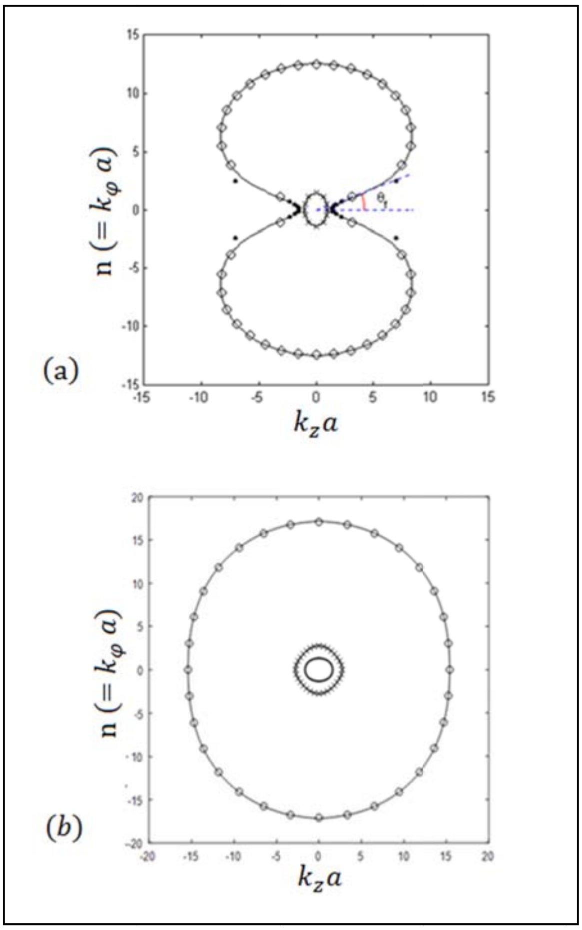

Figure 5. Comparison between of the exact dispersion curves represented by $D = 0$(-) and the approximate dispersion curves represented by $D_{mem1} = 0(\cdot)$,$D_{mem2} = 0$ (x) and $D_{flex} = 0$ (∘) at (a) 9,238 *Hz* (= 0.84 *Ω*) and (b) 18,275 *Hz* (= 1.65 *Ω*).

The characteristics of the dispersion relation in Figure 4 are more clearly identified by comparing the exact dispersion curves represented by $D = 0$ with the approximate dispersion curves represented by $D_{mem1} = 0$, $D_{mem2} = 0$ and $D_{flex} = 0$ in Figure 5. In Figure 5(a) the flexural waves are clearly identified as small circles on the top and bottom portions of the figure 8 curve which are defined by propagation angle θ_k defined by

$$|\theta_f| \leq |\theta_k| \leq |\pi - \theta_f| \tag{25}$$

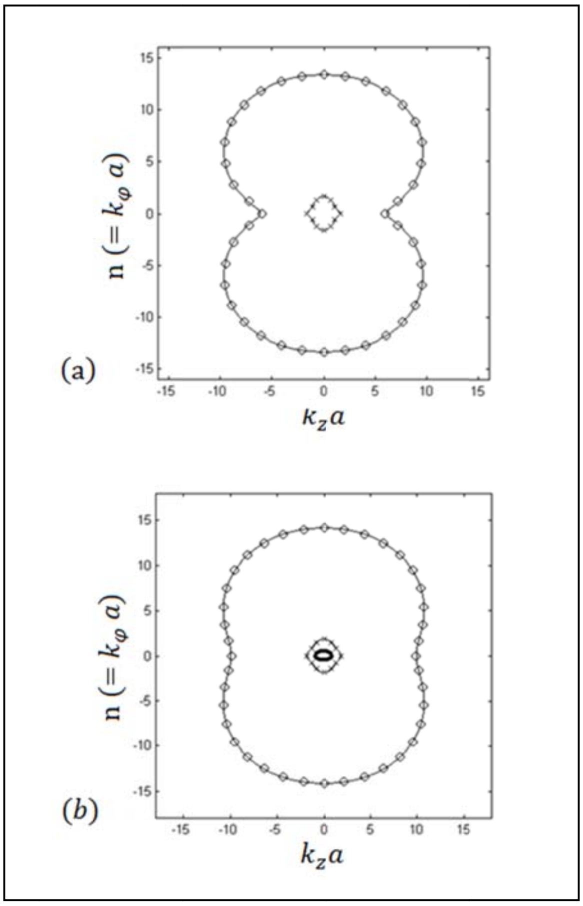

Figure 6. Comparison between of the exact approximate dispersion curves represented by $D = 0$ (-) and the approximate dispersion curves represented by $D_{mem1} = 0$ (·), $D_{mem2} = 0$ (x) and $D_{flex} = 0$ (∘) at (a) 10,809 *Hz* (= 0.98 *Ω*) and (b) 12,133 *Hz* (= 1.1 *Ω*).

where $\theta_f = cos^{-1}\{\frac{\Omega^2}{1-\nu^2}\}^{\frac{1}{4}}$ (= 20.7 ° for $\Omega = 0.84$ and $\nu = 0.28$) is determined from equation 7. The dispersion curves representing $D_{mem1} = 0$ for membrane waves form the center of the figure-8. Those curves form two open curves which approximately resemble parabolas. Those waves have characteristics for propagation near the axial direction: shear waves below $\Omega_m (= 0.94$ for $\nu = 0.28)$, and longitudinal waves between $\Omega_m (= 0.94$ for $0.28)$ and $\Omega_f (= 0.96$ for $\nu = 0.28)$. In other directions up to the direction defined by $|\theta_k| \leq \theta_f$, those are associated with membrane waves whose phase velocities are much smaller than the speeds of either longitudinal or shear waves propagating in the axial direction. Above the ring frequency the waves governed by $D_{mem1} = 0$ correspond to longitudinal waves.

The dispersion curve representing $D_{mem2} = 0$ forms an inner closed curve. Those waves have characteristics for propagation near the axial direction: longitudinal waves below $\Omega_m (= 0.94$ for $\nu = 0.28)$, and shear waves above Ω_m. Above the ring frequency $(\Omega = 1)$, the waves governed by $D_{mem2} = 0$ correspond to shear waves, respectively.

Figure 6(a) and (b) show the characteristics of the dispersion relations at dimensionless frequency $\Omega = 0.98$ and $\Omega = 1.1$ just below and above the ring frequency $(\Omega = 1)$, respectively. Figure 6(a) shows that the dispersion curve representing $D_{mem1} = 0$ does not exist between Ω_f $(= 0.96$ for $\nu = 0.28)$ and the ring frequency ($\Omega = 1$).

When waves spread with energy out from the source point over a plate made of homogeneous isotropic elastic material, the wave fronts form circles. The waves propagating on the cylindrical shell made of homogeneous isotropic elastic material resembles the waves propagating on the plate quite higher than the ring frequency. However, the waves on the shell have the anisotropic nature of propagation at low frequencies up to frequencies somewhat higher than the ring frequency. The figure-8 of the dispersion curve is one of typical examples which shows the anisotropic nature of propagation. The outward normal direction to the dispersion curve at any given point gives the group velocity direction, or the direction at which energy flows, for a wave with the corresponding wavenumber and phase velocity direction. Thus, the less circular the wavenumber curve becomes, the more anisotropic the wave propagation on the shell are. The further detailed research about the anisotropic characteristics of waves on the cylindrical shells is underway.

5. Conclusions

In this paper, the characteristics of elastic wave propagation on the shell around the ring frequency have been analyzed theoretically and experimentally. The assumption of a thin shell allowed the dispersion relation to be separated into three relations related to the propagation of flexural waves and two types of membrane waves. Those relations have been used to identify the characteristics of the dispersion curves which have been obtained theoretically and experimentally. The dispersion curves correspond to two and three curves below and above the ring frequency, respectively. Above the ring frequency, three dispersion curves have been clearly identified to be those of flexural, shear and longitudinal waves, respectively. Below the ring frequency, the characteristics of two dispersion curves have been identified with dependency of the direction of wave propagation. Those results have been effectively used to identify the characteristics of waves propagation on the cylindrical shell around the ring frequency.

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