

# Modelling space-variant reflections in wavenumber array spectra

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## Summary

This paper introduces and investigates a novel single-layer Fourier-based near-field acoustic holography (NAH) technique suited to measurements with a linear microphone array in the presence of a reflector that is perpendicular to the array. The general idea is to model the reflected plane waves as weighted versions of the incident waves. Numerical experiments are performed in order to test the reconstructions against the conventional (free-field) NAH technique and reference results obtained from theoretical calculations. The most important outcome of this paper is that the reconstruction errors decrease as the wavelength of the plane waves is an integer multiple of the array length, and this is attributed to spectral leakage and windowing artefacts introduced by the use of the spatial Fourier transform. In the case of greatest leakage, that is, the wavenumbers of the plane waves are exactly in the middle of two adjacent bins of the wavenumber domain, the errors can exceed 100%. In the case of no leakage, the errors are no greater than 25%. Overall, this study points towards further investigation of the method.

PACS no. 43.20.Gp, 43.60.Ac, 43.60.Fg, 43.60.Sx

# 1. Introduction

Fourier-based near-field acoustic holography was first introduced in the '80s [1], and it comprises an efficient technique to reconstruct the pressure or the velocity field of a given source, by means of microphone array measurements in the near-field of such a source. One of the typical limitations of the original NAH formulation is the lack of a model of the acoustic environment in which the measurements are performed. In other words, the presence of reflecting surfaces, or of additional sources other than the target source, can more often than not impact the reconstruction accuracy significantly.

A number of studies have attempted to filter out the reflections, and the majority rely on array measurements in two planes (or layers). Examples of these are applications of the statistically optimized NAH [2, 3, 4, 5, 6, 7], the equivalent source method [8, 9, 10, 11, 12], and the Fourier-based methods [13, 14, 15, 16]. As opposed to double-layer formulations, Zea and Arteaga have proposed a single-layer extension of Fourier-based NAH to account for the presence of a reflecting surface that is parallel to the microphone array [17].

Yet another limitation of Fourier-based NAH is that the reconstruction and measurement planes must be parallel, which guarantees that the spatial convolutions performed in the Kirchhoff-Helmholtz (or Rayleigh) integrals can be solved in the wavenumber domain by means of spatial Fourier transforms [18]. This is in fact another reason why the method in [17] can be formulated in the wavenumber domain. In this paper, however, we consider the presence of a reflector that is perpendicular to the array, thus the convolution operators describing the propagation paths from the array to the reflector are no longer spaceinvariant. Although there is room for application of boundary element methods, such as an inverse WRW model [19, 18], there are no approaches, to the author's knowledge, of single-layer Fourier-based NAH methods that can tackle this particular problem.

This paper introduces and examines a new singlelayer Fourier-based NAH formulation, which can be used to remove the sound field due to a reflector that is perpendicular to a linear microphone array. The general idea is to represent the reflected plane waves as weighted incident waves on the reflector. Numerical experiments with synthetic plane waves are performed in order to test the accuracy of the method against that of free-field NAH [20]. The results reveal that the method is most accurate whenever the wavelength of the plane waves is an integer multiple of the array

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length, and this is attributed to the leakage due to the use of spatial Fourier transforms.

# 2. Background

This section includes the pertinent theoretical background to put the present work into context. The section begins with a description of the WRW model and the definition of space-variance of acoustic wave propagation. Thereafter, we include a section with the concept of spatial Fourier transform and the interpretation of sound fields in terms of plane waves. Lastly, the concept of plane-wave specular reflections from locally reacting surfaces is briefly discussed. Throughout the remainder of the text the time-harmonic dependence of the fields  $e^{j\omega t}$  is omitted. In addition, vectors and matrices are represented with bold-faced lower-case and upper-case letters respectively.

### 2.1. WRW model

The acoustic problem is illustrated in Figure 1. The pressure field in the source line propagates to the array in the form of direct field, and adds with the reflected field that comes from the reflector. The reflector is located at a distance  $\delta$  from the right-most microphone position, and the spacing between the microphones is  $\Delta x$ . The array and the source lines are located at  $z = z_h$  and  $z_s$  respectively. Then, the total



Figure 1. Geometrical illustration of the acoustic problem.

sound field at the array via the WRW model reads [19]

$$\mathbf{p}_h = \left(\mathbf{W}_{sh} + \mathbf{W}_{rh} \mathbf{R} \mathbf{W}_{sr}\right) \mathbf{p}_s,\tag{1}$$

where  $\mathbf{p}_i$  is a column vector with the pressure field at the line  $z = z_i$ , and the matrix **R** contains the reflectivity impulse responses of the reflector [19, 21]. In addition, any matrix  $\mathbf{W}_{nm}$  contains in its columns the relevant Dirichlet Green's functions that represent the acoustic propagation between the line sets n and m [19]. In mathematical terms, a column of  $\mathbf{W}_{nm}$  is defined as

$$\mathbf{w}_{nm} = -jk\Delta x \frac{z_m - z_n}{2r_{nm}} H_1^{(2)}(kr_{nm}), \qquad (2)$$

where  $r_{nm} = \sqrt{(x_m - x_n)^2 + (z_m - z_n)^2}$ ,  $j^2 = -1$ , k is the acoustic wavenumber and  $H_1^{(2)}$  is the first-order Hankel function of the second kind [22].

It can be shown that  $\mathbf{W}_{nm}$  exhibits Toeplitz structure whenever the line sets n and m are parallel to each other [18], because the convolution operators are space-invariant and the convolution theorem holds [23]. In that case, the convolution can be solved in the wavenumber domain by means of using the operator

$$\mathbf{G}_{nm} = \operatorname{diag} \left\{ e^{j\mathbf{k}_z(z_m - z_n)} \right\},\tag{3}$$

where  $\mathbf{k}_{z}$  is defined as

$$\mathbf{k}_{z} = \begin{cases} \sqrt{k^{2} - \mathbf{k}_{x}^{2}}, & \mathbf{k}_{x} \leq k, \\ j\sqrt{\mathbf{k}_{x}^{2} - k^{2}}, & \mathbf{k}_{x} \succ k, \end{cases}$$
(4)

and  $\mathbf{k}_x$  is the vector containing the wavenumbers (or spatial frequencies) sampled by the array. It should be noted that a real-valued  $\mathbf{k}_z$  corresponds to a propagating wave, whereas an imaginary-valued  $\mathbf{k}_z$  to an evanescent wave [20].

However, as it is the case in Figure 1, the line sets s and r, as well as r and h, are perpendicular, which introduces space-variance in the convolution operators and Eq. (3) can no longer be used.

## 2.2. Spatial Fourier transforms

The use of spatial Fourier transforms in acoustics brings a powerful tool to the analysis of sound fields in terms of plane waves and their incidence angles [20]. For example, a plane wave that propagates with an angle  $\theta$  with respect to the z axis is defined as:

$$p(x,z) = e^{-j(kx\sin\theta + kz\cos\theta)}.$$
(5)

If z is constant, we can apply the spatial Fourier transform along the x axis as

$$\tilde{p}(k_x) = \int_{-\infty}^{\infty} p(x)e^{-jk_x x} dx =$$
$$= e^{-jkz\cos\theta}\delta(k_x - k\sin\theta), \quad (6)$$

where  $k_x$  is the spatial frequency of the field in the x direction. In other words, the plane wave can be represented in the wavenumber domain as a Dirac delta centered at the wavenumber  $k_x = k \sin \theta$ .

In practice, the measured pressure field is finite and discrete, thus the application of spatial Fourier transforms can lead to wavenumber leakage and other distortions due to spatial windowing [20]. For instance, if a rectangular window is used, then the wavenumber spectrum is

$$\tilde{\mathbf{p}}(\mathbf{k}_x) = e^{-jkz\cos\theta}\operatorname{sinc}\left[(\mathbf{k}_x - k\sin\theta)\frac{L}{2}\right],\quad(7)$$

where  $\operatorname{sinc}(\mathbf{x}) = \operatorname{sin}(\mathbf{x})/\mathbf{x}$ , and *L* is the array length [20].

#### 2.3. Plane-wave reflections

The theory of specular reflections from locally reacting surfaces comprises an initial approximation to the problem of plane-wave reflection. A locally-reacting surface is that in which the pressure (and particle velocity) at a given point in the surface depends only on the pressure (particle velocity) at that point [22]. In order to illustrate this, let us consider the geometry in Figure 1, with the reflector positioned at x = 0. Further assume that the reflector is considerably larger than the acoustic wavelength. Then, for some  $A_{\rm inc} \in \mathbb{C}$ , the incident and reflected plane waves read

$$p_{\rm inc}(x,z) = A_{\rm inc} e^{-j(kx\sin\theta + kz\cos\theta)},$$
(8a)

$$p_{\rm ref}(x,z) = CA_{\rm inc}e^{-j(-kx\sin\theta + kz\cos\theta)}.$$
 (8b)

Then, at the line x = 0, the following relationship holds [22]

$$p(0,z) = p_{\rm inc}(0,z) + p_{\rm ref}(0,z) = (1+C)p_{\rm inc}(0,z).$$
(9)

Next, we can determine the particle velocity via application of Euler's equation of motion to Eqs. (8a)-(8b) as follows

$$\begin{aligned} u_{\rm inc}(x,z) &= -j \frac{1}{Z_0 k} \frac{\partial}{\partial x} p_{\rm inc}(x,z) = -\frac{\sin \theta}{Z_0} p_{\rm inc}(x,z), \\ (10a) \\ u_{\rm ref}(x,z) &= -j \frac{1}{Z_0 k} \frac{\partial}{\partial x} p_{\rm ref}(x,z) = \frac{\sin \theta}{Z_0} p_{\rm ref}(x,z), \\ (10b) \end{aligned}$$

where  $Z_0$  is the specific impedance of air. Thus, the total particle velocity at the line x = 0 follows

$$u(0, z) = u_{\rm inc}(0, z) + u_{\rm ref}(0, z) =$$
  
= -(1 - C)p\_{\rm inc}(0, z). (11)

Lastly, the surface impedance can be found via

$$Z = -\frac{p(0,z)}{u(0,z)} = \frac{Z_0}{\sin\theta} \frac{1+C}{1-C}.$$
 (12)

Then, the plane-wave reflection coefficient reads in the discrete wavenumber domain

$$\mathbf{C} = \operatorname{diag}\left\{\frac{\mathbf{k}_x - k\beta}{\mathbf{k}_x + k\beta}\right\},\tag{13}$$

where  $\beta$  is the specific acoustic admittance of the reflector.

## 3. Separation method

Let us assume that an array of pressure receivers is located as shown in Figure 1. The following equations are all written in the wavenumber domain, and this shall be denoted with a tilde above the pressure fields. The total sound field measured at the array (hologram) line equals the sum of the direct field and the reflected field

$$\tilde{\mathbf{p}}_h = \tilde{\mathbf{p}}_{\rm df} + \tilde{\mathbf{p}}_{\rm rf},\tag{14}$$

with the reflected field modelled as a function of the direct field as follows

$$\tilde{\mathbf{p}}_{\rm rf} = \mathbf{M} \boldsymbol{\Psi} \tilde{\mathbf{p}}_{\rm df}.\tag{15}$$

Here the permutation matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{J} & \mathbf{0} \end{pmatrix},\tag{16}$$

where **J** is an anti-diagonal identity matrix of appropriate dimensions, and the reflection filter  $\Psi = \text{diag}\{\psi\}$  has its diagonal elements defined as

$$\psi = \begin{cases} \frac{\mathbf{k}_x - k\beta}{\mathbf{k}_x + k\beta} e^{-j\mathbf{k}_x(L+2\delta)}, & \mathbf{k}_x \leq k\\ 0, & \mathbf{k}_x \succ k, \end{cases}$$
(17)

where the notation  $\leq$  and  $\succ$  denote point-wise vector inequalities.

If the reflector is located on the left-hand side of the array, then  $\tilde{\mathbf{p}}_{rf} = (\mathbf{M} \boldsymbol{\Psi})^H \tilde{\mathbf{p}}_{df}$ , where H denotes the Hermitian matrix transpose. Note that  $\boldsymbol{\Psi}$  discriminates evanescent waves, assuming they do not interact with the reflector. Propagation towards the source line  $z = z_s$  entails the following modifications to the system of equations

$$\tilde{\mathbf{p}}_h = \mathbf{G}_R \tilde{\mathbf{p}}_s,\tag{18}$$

where

$$\mathbf{G}_{R} = \left[\mathbf{Q} + \mathbf{G}_{hs}^{2}\mathbf{M}\boldsymbol{\Psi}\right]\mathbf{G}_{sh}.$$
 (19)

Here  $\mathbf{Q} = \text{diag}\{q\}$  has its elements defined as

$$q = \begin{cases} e^{-2j\mathbf{k}_z(z_h - z_s)}, & \mathbf{k}_x \prec 0\\ 1, & \mathbf{k}_x \succeq 0. \end{cases}$$
(20)

If the reflector is on the left-hand side of the array, then **Q** must be premultiplied with  $\mathbf{G}_{sh}^2$ .

The problem in Eq. (18) is ill-posed and sensitive to noise in the measurements  $\tilde{\mathbf{p}}_h$ , thus a regularization strategy must be adopted. In this paper we employ a general Tikhonov filter approach via the singular value decomposition [24], which consists of minimizing the functional

$$\mathcal{F}(\tilde{\mathbf{p}}_s) = \|\tilde{\mathbf{p}}_h - \mathbf{G}_R \tilde{\mathbf{p}}_s\|_2^2 + \mu^2 \|\mathbf{\Gamma} \mathbf{V}^H \tilde{\mathbf{p}}_s\|_2^2, \quad (21)$$

where  $\Gamma$  is the Tikhonov matrix,  $\mu > 0$  is the regularization parameter, and the singular value decomposition of  $\mathbf{G}_R$  reads

$$\mathbf{G}_R = \mathbf{U} \boldsymbol{\Sigma} \mathbf{V}^H, \tag{22}$$

where  $\Sigma$  is a diagonal matrix whose elements are the singular values, here denoted with  $\sigma$ . This allows us then to find the global minimum of Eq. (21) with the Tikhonov filter **F** via [24]

$$\tilde{\mathbf{p}}_{s}^{(\text{rec})} = \mathbf{V} \mathbf{F} \boldsymbol{\Sigma}^{-1} \mathbf{U}^{H} \tilde{\mathbf{p}}_{h}, \qquad (23)$$

where

$$\mathbf{F} = \left[ \mathbf{\Sigma}^H \mathbf{\Sigma} + \mu^2 \mathbf{\Gamma}^H \mathbf{\Gamma} \right]^{-1} \mathbf{\Sigma}^H \mathbf{\Sigma}, \qquad (24)$$

and

$$\boldsymbol{\Gamma} = \mu^2 \left[ \boldsymbol{\Sigma}^H \boldsymbol{\Sigma} + \mu^2 \right]^{-1}.$$
 (25)

Adequate choice of  $\mu$  can be done, for example, with the generalized cross-validation [25], the L-curve [26] or the Morozov's discrepancy principle [27]. In this paper we use the L-curve.

## 4. Numerical investigation

## 4.1. Setup

The simulation setup is illustrated in Figure 1. The distance between the array and source lines is  $z_h$  –  $z_s = 6$  cm, and the distance from the array to the reflector is  $\delta = 9$  cm. The reflector is deemed rigid, that is,  $\beta = 0$ . The frequency range of interest is from 500 Hz to 2500 Hz in steps of 100 Hz. The sound fields are composed of 20 plane waves propagating in the +xdirection, and 10 plane waves propagating in the -xdirection, all with random complex amplitudes. The angles of incidence are chosen such that the wavenumbers of the plane waves are exactly sampled by the array. Eqs. (8a)-(8b) are then used to synthesize the relevant sound fields at the array and source lines. An array of 50 microphones spaced by 5 cm is used. Complex Gaussian noise is added to the measurement line  $\mathbf{p}_h$  such that the signal-to-noise ratio is 25 dB. Freefield NAH solutions are regularized with the modified Tikhonov filter described in [28].

#### 4.2. Evaluation metrics

In order to quantify the performance of the reconstruction, the following error metric is used

$$\varepsilon = \frac{\|\mathbf{p}_{s}^{(\text{ref})} - \mathbf{p}_{s}^{(\text{rec})}\|_{2}}{\|\mathbf{p}_{s}^{(\text{ref})}\|_{2}} \cdot 100\%, \tag{26}$$

where  $\|\cdot\|_2$  is the  $\ell_2$  norm, superscripts (ref) and (rec) denote reference and reconstructed solutions. In addition, a reflected-to-direct ratio (RDR) is estimated at the measurement line  $z = z_h$  via

$$RDR = \frac{\|\mathbf{p}_{rf}\|_2}{\|\mathbf{p}_{df}\|_2} \cdot 100\%.$$

$$(27)$$

In essence, this quantity indicates what percentage of the measured sound field corresponds to reflections, with respect to the direct field, as well as how accurate will the reconstruction be via the free-field NAH method.

## 4.3. Reconstruction results

The reconstruction errors of the separation method and the free-field NAH method are shown in Figure 2, together with the RDR. The errors via free-field NAH indicate that the reconstructions are largely disturbed by the reflections, and this can be explained with such high RDR percentages. On the contrary, the reconstruction errors via the proposed method appear to be no greater than 30%.



Figure 2. Reconstruction errors via the separation method and the free-field NAH method, as well as the reflectedto-direct ratio (RDR).

An illustration of the reconstruction procedure is shown in Figure 3, where the left-hand side shows wavenumber spectra and the right-hand side shows space-domain pressure. As it can be seen in Figure (3a), the reconstruction via free-field NAH seems to accurately recover the negative axis of the wavenumber spectrum, but it fails to recover the positive axis which is distorted by the reflected plane waves. On the other hand, the reconstruction via the separation method appears to be accurate within the whole wavenumber domain, and this can also be confirmed in Figure (3b).

### 4.4. Influence of spectral leakage

In this section we examine how the reconstruction errors vary with plane waves whose wavenumbers are not exactly sampled by the array. To do so, we define a quantity  $\zeta$  that represents the percentage away from an exact bin in the wavenumber domain. In principle, the influence of this quantity is greatest at  $\zeta = 50\%$ , and is smallest as the wavenumber of the plane waves gets closer to a wavenumber bin. For the sake of brevity we include results at 1500 Hz, varying  $\zeta$  from 0% to 100% in steps of 5%. Furthermore, in an attempt to account for the leakage, we include reconstruction errors using a reflection filter  $\Psi$  with modified phase  $(\mathbf{k}_x + \zeta 2\pi/L)(L + 2\delta)$  for  $\mathbf{k}_x \leq k$ .

Figure 4 shows the reconstruction errors versus  $\zeta$  for the free-field NAH method, the separation method, and the separation method with modified filter phase. The errors obtained with free-field NAH are nearly invariant with respect to  $\zeta$ , whereas those obtained with the separation method, as expected, increase as  $\zeta$  tends to 50% and decrease as  $\zeta$  tends to 0% or 100%.



Figure 3. Sound pressure level (SPL) plots in dB re 20  $\mu$ Pa at 1500 Hz in (a) the wavenumber domain, and (b) the spatial domain. The notation used is: reference solution  $\mathbf{p}_s^{(\text{ref})}$ , reconstruction via the separation method  $\mathbf{p}_s^{(\text{rec,sm})}$ , via the free-field NAH method  $\mathbf{p}_s^{(\text{rec,ff})}$ , as well as the measurement line  $\mathbf{p}_h$ . The reconstruction error at this frequency is 15.59% for the separation method, and 158.2% for free-field NAH.

As regards the separation method with modified filter phase, there appears to be a substantial decrease in reconstruction error. Although in practice there is no knowledge of  $\zeta$ , it is interesting to see that the reconstructions can still be accurate if this parameter is taken into account in the model.



Figure 4. Reconstruction errors versus  $\zeta$  at 1500 Hz for the free-field NAH method, the separation method, and the separation method with modified filter phase.

# 5. Concluding remarks

This paper introduces a new Fourier-based near-field acoustic holography (NAH) method that accounts for the presence of a reflector that is perpendicular to a linear microphone array. The main novelty is that the formulation of the method is derived in the wavenumber domain, in spite of the difficulties related to the space-variant nature of the propagating functions. The regularized inversion is done by means of Tikhonov filtering via the singular value decomposition. Numerical experiments are included in order to investigate the performance of the method against the conventional (free-field) NAH method.

In general the performance of free-field NAH is strongly disturbed by the reflections and yields inaccurate reconstructions. On the other hand, the separation method is most accurate when the synthetic sound fields are composed of plane waves which are exactly sampled by the array –i.e. minimum leakage. As an initial investigation, the influence of spectral leakage in the reconstructions has been examined, and it has been shown that the separation method can still provide accurate results if the leakage is included in the model. It should however be stressed that in general it is likely that some plane waves have more leakage than others, thus the results shown here do not necessarily apply to practical situations.

## 6. Future work

There are three directions for future work that can be followed from this paper. The first is to seek for a generalized model of spectral leakage, such that the separation method is independent of  $\zeta$ . Another study can be done to extend the formulation to two-dimensional planar arrays, which should in principle be straightforward. Lastly, the performance of the method in the presence of partially absorbing reflectors can be investigated by means of accounting for the corresponding admittance in the reflection filter  $\Psi$ .

#### Acknowledgement

This work is funded by the Swedish Research Council (Vetenskapsrådet), under grant No. 2016-04366.

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