Transient solutions for directional responses of parabolic transmitters and receivers

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Summary
The impulse response is obtained for the directional dependence in the far field of a parabolic transmitter formed by point source located at the focus of a paraboloidal reflector. The impulse response at the focus of a parabolic receiver as a function of the direction from which a plane wave is incident on a paraboloidal reflector is obtained via reciprocity. The corresponding step response is convolved with the time derivative of the incident waveform to obtain the general transient response at the focus of a parabolic receiver.

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1. Introduction

Parabolic reflectors are employed in acoustics and electromagnetics as both transmitters and receivers to concentrate energy coherently when the wavelength is short compared with the size and minimum radius of curvature of the reflector. In the case of a transmitter, a spherical wavefront emanating from a point source at the focus of the paraboloid becomes approximately planar following reflection and radiates away as a collimated beam. For a receiver, an incident planar wavefront propagating along the axis of the paraboloid is converted following reflection into an approximately spherical wavefront that converges at the focus of the paraboloid.

Analyses of paraboloidal reflectors are traditionally performed in the frequency domain; see especially Wahlström [1] for a frequency-domain analysis of acoustical parabolic receivers and discussion of earlier work. The present work is performed in the time domain. In a previous time-domain analysis of the acoustical transmitter, a transient solution along the axis of a paraboloidal reflector was obtained in terms of an analytical expression for the impulse response [2]. As in the case of an ellipsoidal reflector [3], the axial impulse response for a paraboloidal reflector consists of three terms, a pair of Dirac delta functions corresponding to the beginning and end of the reflected time waveform, and a third term, a continuous distribution in time referred to as the wake, connecting the two delta functions. Tsai et al. [4] subsequently employed this approach to obtain a transient axial solution for the parabolic receiver.

Presented here is a transient solution for the pressure at the focus of a parabolic receiver in terms of a step response that accounts for the propagation direction of an incident plane wave. The analysis begins by extending the transient axial solution for a parabolic transmitter [2] to obtain the angular dependence of the impulse response for the reflected pressure waveform in the far field. In the limit of a shallow reflector the expression for the pressure in the far field reduces to the result obtained by Morse [5] for transient radiation from a circular piston. The directional impulse response for a parabolic receiver is then obtained via reciprocity from the far-field impulse response for the transmitter. The corresponding step response is convolved with the time derivative of the incident waveform to obtain the general transient response at the focus of a parabolic receiver.

2. Far field of parabolic transmitter

A point source is located at the focus of a paraboloidal reflector that is symmetric about the z axis with vertex at $z = 0$, focus at $z = z_f$, and aperture in the plane $z = d$. If a spherical wave emanating from the focus is incident on the vertex with pressure $p_0 f(t)$, then the reflected pressure p_{far} in the far field at spherical coordinates (r, θ) is given by

$$\frac{p_{\rm far}(r,\theta,\tau)}{p_0} = \int_{-\infty}^{\infty} h_{\rm far}(r,\theta,\tau') f(\tau - \tau')\, d\tau' \quad (1)$$

where $\tau = t - r/c_0$ is retarded time, θ is the polar angle with respect to the z axis,

$$h_{\rm far}(r,\theta,\tau) = \frac{c_0}{2\pi r}\int_0^{d/z_f} w(\zeta,\theta,\tau)\frac{d\zeta}{1+\zeta} \quad (2)$$

is the impulse response, and the angular dependence is taken into account by the dimensionless function

$$w(\zeta,\theta,\tau) = \frac{[(c_0\tau/2z_f) - \zeta\sin^2(\theta/2)]\,{\rm rect}(\tau;\tau_-,\tau_+)}{\{\zeta\sin^2\theta - [(c_0\tau/2z_f) - \zeta\sin^2(\theta/2)]^2\}^{3/2}} \quad (3)$$

where

$$\tau_\pm = (2z_f/c_0)[\zeta\sin^2(\theta/2) \pm \zeta^{1/2}\sin\theta] \quad (4)$$

The rectangle function ${\rm rect}(\tau;\tau_-,\tau_+)$ is defined to be unity for $\tau_- \le \tau \le \tau_+$ and zero otherwise.

As in previous work [1, 2, 3, 4], the solution is based on the use of geometrical acoustics to establish the boundary condition for the pressure on the surface of the reflector. Geometrical acoustics suffices when the wavelength is short compared with the minimum radius of curvature of the surface, corresponding to $kz_f \gg 1$ where k is the characteristic wavenumber of the pressure field. Integration over the surface is then performed using the far-field approximation of the Green's function.

The solution on axis is obtained by recognizing that

$$\lim_{\theta\to 0} w(\zeta,\theta,\tau) = (4\pi z_f^2/c_0^2)\delta'(\tau) \quad (5)$$

where δ is the Dirac delta function, such that substitution of Eq. (2) in Eq. (1) yields

$$\frac{p_{\rm far}(r,0,\tau)}{p_0} = \frac{2z_f^2}{c_0 r}\ln\left(1+\frac{d}{z_f}\right)f'(\tau) \quad (6)$$

The primes on the functions $\delta(\tau)$ and $f(\tau)$ signify the derivative with respect to the argument. In contrast with the axial solution in the near field [2], the impulse response does not separate into three distinct terms, two delta functions and a wake. Along the axis in the far field the three contributions coalesce into the time derivative of a delta function in Eq. (5), and off axis anywhere, near field as well as far field, the lack of symmetry in the arrivals from the reflector precludes such a decomposition.

Arriving ahead of the reflected wave in the far field is the wave propagating directly from the point source. The direct wave arrives in advance of the reflected wave by the time interval

$$\Delta t = \frac{2}{c_0}\left[z_f^{1/2}\cos(\theta/2) - d^{1/2}\sin(\theta/2)\right]^2 \quad (7)$$

The difference in arrival times is a maximum on axis, $\theta = 0$, for which $\Delta t = 2z_f/c_0$. In addition to time delay, the axial amplitude of the reflected wave exceeds that of the direct wave by a factor of $\sim 2kz_f\ln(1 + d/z_f)$.

The far-field expression for reflection of a plane wave from a circular disk of radius a can be obtained from Eqs. (1)–(4) by taking $d \to 0$ such that the reflector is planar, and $z_f \to \infty$ such that the incident wavefront is planar over the surface of the reflector. The limits are taken in a way that maintains a finite value for the radius of the aperture of the paraboloidal reflector, which is given by $a = 2(dz_f)^{1/2}$. Equations (2)–(4) then yield for the impulse response

$$\lim_{\substack{d\to 0\\ z_f\to\infty}} h_{\rm far}(r,\theta,\tau) = -\frac{c_0^2\tau\,{\rm rect}(\tau;\tau_-^{\rm disk},\tau_+^{\rm disk})}{\pi r\sin^2\theta\sqrt{a^2\sin^2\theta - c_0^2\tau^2}} \quad (8)$$

where $\tau_\pm^{\rm disk} = \pm(a/c_0)\sin\theta$. Substitution into Eq. (1) and defining $U(t) = (p_0/\rho_0 c_0)f(t)$ to be the effective velocity of the circular disk yields Eq. (28.12) of Morse [5] for transient radiation in the far field of a baffled circular piston (apart from an evidently mistaken factor of 2 in the denominator of Morse's expression).

Plots of $w(\zeta,\theta,\tau)$ as functions of τ are presented in Fig. 1 for $\theta = 30°$ and three values of $\zeta = z/z_f$ corresponding to reflections from different circular rings on the surface of the paraboloid formed by the intersections of planes perpendicular to the z axis. For an impulse $f(\tau) = t_0\delta(\tau)$ incident on a ring with small axial width $\Delta\zeta = \Delta z/z_f$ the pressure $\Delta p_{\rm far}$ in the far field is

$$\Delta p_{\rm far} = p_0\frac{c_0 t_0}{2\pi r}w(\zeta,\theta,\tau)\frac{\Delta\zeta}{1+\zeta} \quad (9)$$

The impulse response in Eq. (2) is the sum of the waveforms reflected from every such ring from the vertex ($\zeta = 0$) to the aperture ($\zeta = d/z_f$) of the reflector. For field points off axis ($\theta \ne 0$), the duration $T = \tau_+ - \tau_- = (4z_f/c_0)\zeta^{1/2}\sin\theta$ of the signal increases with ζ according to the limits in Eq. (4) because the rings increase in diameter, and the time between the first and last arrival of an impulse reflected from successively larger rings increases accordingly. Similarly, for a given ring (value of ζ) T increases as θ increases.

3. Focus of parabolic receiver

The reflected pressure produced at the focus of a parabolic receiver by an incident plane wave propagating at angle θ with respect to the z axis is obtained by applying the principle of reciprocity to the far-field solution for the transmitter. Thus suppose the point source for the transmitter is moved from the focus to a location (r, θ) in the far field. The point source

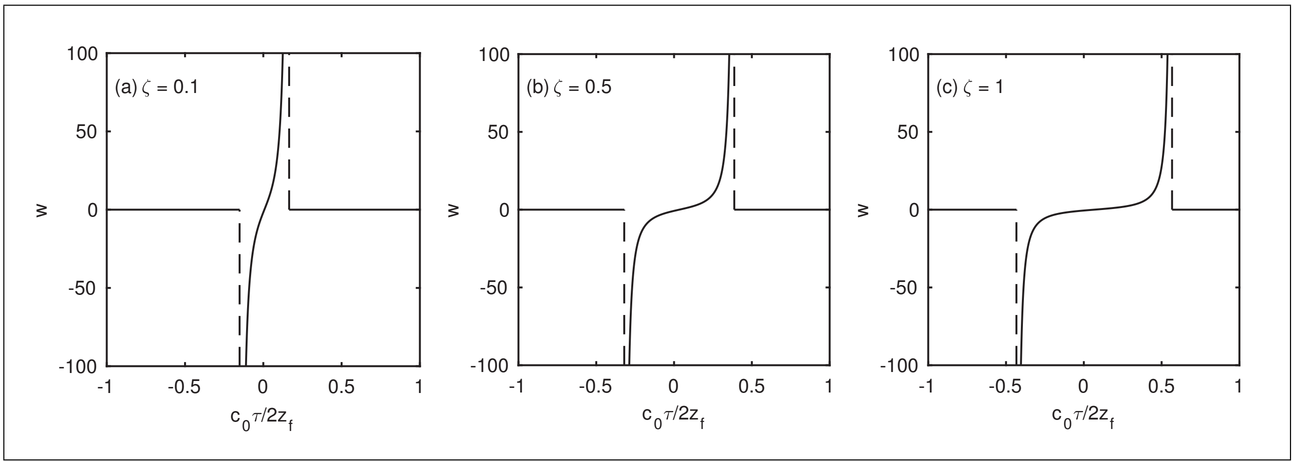

Figure 1. Time dependence of the function $w(\zeta,\theta,\tau)$ in Eq. (3) for $\theta = 30^\circ$ and (a) $\zeta = 0.1$, (b) $\zeta = 0.5$, and (c) $\zeta = 1$.

for the transmitter produces pressure $p_0 f(t)$ at distance z_f, but we want the point source to produce the same pressure $p_0 f(t)$ at the vertex of the reflector, which is now at distance r rather than distance z_f from the point source. The amplitude of the spherical wave emanating from the point source must therefore be increased by the factor r/z_f. Since the solution is linear, Eq. (2) is scaled by the same factor to obtain

$$h_{\rm foc}(\theta,\tau) = \frac{c_0}{2\pi z_f}\int_0^{d/z_f} w(\zeta,\theta,\tau)\frac{d\zeta}{1+\zeta} \tag{10}$$

for the impulse response of a point receiver at the focus, where $\tau = t - z_f/c_0$. Because r is arbitrarily large, the incident wave may be considered planar in the vicinity of the reflector. Equation (10) is therefore the desired impulse response for a plane wave incident on the receiver at angle θ, with Eqs. (3) and (4) defining $w(\zeta,\theta,\tau)$ as before.

The pressure at the focus is

$$\frac{p_{\rm foc}(\theta,\tau)}{p_0} = \int_{-\infty}^{\infty} h_{\rm foc}(\theta,\tau')f(\tau-\tau')\,d\tau' \tag{11}$$

Since the definition of $w(\zeta,\theta,\tau)$ is unaltered, the limit in Eq. (5) applies, and the pressure at the focus for a plane wave at normal incidence is the familiar result

$$\frac{p_{\rm foc}(0,\tau)}{p_0} = \frac{2z_f}{c_0}\ln\left(1+\frac{d}{z_f}\right)f'(\tau) \tag{12}$$

For characteristic angular frequency ω and corresponding wavenumber $k = \omega/c_0$ the focusing gain is $G = 2kz_f\ln(1+d/z_f)$. The incident wave arrives at the focus in advance of the reflected wave by the interval of time given by Eq. (7), but for normal incidence its amplitude is less by a factor of $\sim G^{-1}$.

4. Step-response formulation

Alternatives to Eqs. (1) and (11) are convolutions of the unit step response $s(\theta,\tau) = \int_{-\infty}^{\tau} h(\theta,\tau')\,d\tau'$ with the derivative of $f(\tau)$. In place of Eqs. (10) and (11) one obtains for the reflected pressure at the focus a parabolic receiver

$$\frac{p_{\rm foc}(\theta,\tau)}{p_0} = \int_{-\infty}^{\infty} s_{\rm foc}(\theta,\tau')f'(\tau-\tau')\,d\tau' \tag{13}$$

where

$$s_{\rm foc}(\theta,\tau) = \int_0^{d/z_f} v(\zeta,\theta,\tau)\frac{d\zeta}{1+\zeta} \tag{14}$$

is the unit step response and

$$v(\zeta,\theta,\tau) = \frac{\mathrm{rect}(\tau;\tau_-,\tau_+)}{\pi\sqrt{\zeta\sin^2\theta - [(c_0\tau/2z_f) - \zeta\sin^2(\theta/2)]^2}} \tag{15}$$

is a dimensionless function that accounts for the directional dependence of the received waveform, the integral of which over time is a constant, independent of ζ and θ:

$$\int_{-\infty}^{\infty} v(\zeta,\theta,\tau)\,d\tau = 2z_f/c_0 \tag{16}$$

Equations (3) and (15) are related as follows:

$$w(\zeta,\theta,\tau) = \pi\frac{2z_f}{c_0}\frac{\partial}{\partial\tau}v(\zeta,\theta,\tau) \tag{17}$$

For example, Eq. (12) is obtained from the step-response formulation by noting that

$$\lim_{\theta\to 0} v(\zeta,\theta,\tau) = (2z_f/c_0)\delta(\tau) \tag{18}$$

The corresponding step-response formulation for the far field of a parabolic transmitter follows from reciprocity:

$$p_{\rm far}(r,\theta,\tau) = (z_f/r)p_{\rm foc}(\theta,\tau) \tag{19}$$

The formulation in terms of the step response is motivated by examination of Eqs. (3) and (15) for w and

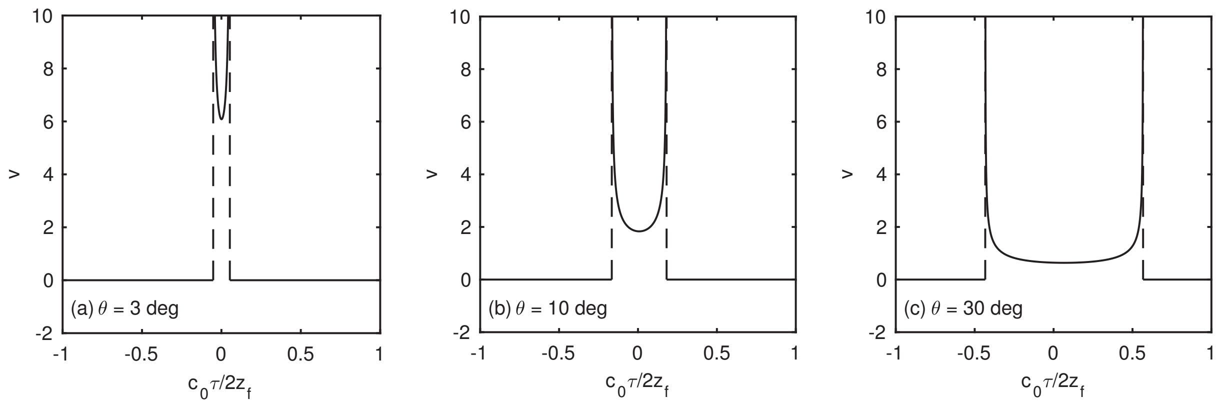

Figure 2. Time dependence of the function $v(\zeta, \theta, \tau)$ in Eq. (15) for $\zeta = 1$ and (a) $\theta = 3°$, (b) $\theta = 10°$, and (c) $\theta = 30°$.

v, respectively. Both functions possess singularities at the temporal points $\tau = \tau_+$ and $\tau = \tau_-$ given by Eq. (4). Letting ϵ characterize the proximity $|\tau - \tau_\pm|$ to either point, in the limit $\epsilon \to 0$ it is observed that the singularities are order $\epsilon^{-3/2}$ in w compared with order $\epsilon^{-1/2}$ in v. The order of the singularities in w prevents integration of this function except in special circumstances, such as in the limit $\theta \to 0$.

Plots of $v(\zeta, \theta, \tau)$ as functions of τ are presented in Fig. 2 for $\zeta = 1$ and three values of θ. Interpreted in the context of a parabolic receiver, each waveform corresponds to the pressure at the focus due to an incident plane wave with time dependence $f(\tau) = u(\tau)$, where $u(\tau)$ is the unit step function, arriving at angle θ and reflecting from a circular ring of small axial width $\Delta\zeta$ on the surface of a paraboloid in the plane determined by ζ:

$$\Delta p_{\text{foc}} = p_0 v(\zeta, \theta, \tau) \frac{\Delta\zeta}{1 + \zeta} \tag{20}$$

As θ becomes small, as in Fig. 2(a), the waveform approaches a delta function according to Eq. (18). At $\theta = 0$ the incident wave reflects from every point on the ring simultaneously. As θ increases, the duration $T = \tau_+ - \tau_-$ of the response increases because the time between when the incident wave reflects from the closest and farthest points on the ring increases, while the area under the curves in Fig. 2 remains constant according to Eq. (16). Note that the parameters used in Figs. 1(c) and 2(c) are the same, and therefore according to Eq. (17) the waveform in Fig. 1(c) is the derivative of that in Fig. 2(c).

5. Conclusion

A time-domain model is provided to examine the directional responses of parabolic transmitters and receivers for arbitrary incident waveforms. More complete results, and derivations of the equations presented here, will appear in a future publication.

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